

Further mathematics
Higher level
Paper 1

Thursday 19 May 2016 (afternoon)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 12]

The set P contains all prime numbers less than 2500.

The set Q is the set of all subsets of P .

(a) Explain why only one of the following statements is true

(i) $17 \subset P$;

(ii) $\{7, 17, 37, 47, 57\} \in Q$;

(iii) $\phi \subset Q$ and $\phi \in Q$, where ϕ is the empty set.

[4]

The set S contains all positive integers less than 2500.

The function $f: S \rightarrow Q$ is defined by $f(s)$ as the set of primes exactly dividing s , for $s \in S$.

For example $f(4) = \{2\}$, $f(45) = \{3, 5\}$.

(b) (i) State the value of $f(1)$, giving a reason for your answer.

(ii) Find $n(f(2310))$.

[4]

(c) Determine whether or not f is

(i) injective;

(ii) surjective.

[4]

2. [Maximum mark: 11]

The lifetime, in years, of a randomly chosen basic vacuum cleaner is assumed to be modelled by the normal distribution $B \sim N(14, 3^2)$.

(a) Find $P\left(B > E(B) + \frac{1}{2}\sqrt{\text{Var}(B)}\right)$. [2]

(b) Find the probability that the total lifetime of 7 randomly chosen basic vacuum cleaners is less than 100 years. [4]

The lifetime, in years, of a randomly chosen robust vacuum cleaner is assumed to be modelled by the normal distribution $R \sim N(20, 4^2)$.

(c) Find the probability that the total lifetime of 5 randomly chosen robust vacuum cleaners is greater than the total lifetime of 7 randomly chosen basic vacuum cleaners. [5]

3. [Maximum mark: 8]

Consider the Diophantine equation $7x - 5y = 1$, $x, y \in \mathbb{Z}$.

(a) Find the general solution to this equation. [3]

(b) Hence find the solution with minimum positive value of xy . [2]

(c) Find the solution satisfying $xy = 2014$. [3]

4. [Maximum mark: 10]

All members of a large athletics club take part in an annual shotput competition. The following data give the distances achieved, in metres, by a random selection of 10 members of the club in the 2016 competition

11.8, 14.3, 13.8, 10.3, 14.9, 14.7, 12.4, 13.9, 14.0, 11.7

The president of the club wishes to test whether these data provide evidence that distances achieved have increased since the 2015 competition, when the mean result for the club was 12.4 m. You may assume that the distances achieved follow a normal distribution with mean μ , variance σ^2 , and that the membership of the club has not changed from 2015 to 2016.

- (a) State suitable hypotheses. [1]
- (b) (i) Give a reason why a t test is appropriate and write down its degrees of freedom.
(ii) Find the critical region for testing at each of the 5% and 10% significance levels. [4]
- (c) (i) Find unbiased estimates of μ and σ^2 .
(ii) Find the value of the test statistic. [3]
- (d) State the conclusions that the president of the club should reach from this test, giving reasons for your answer. [2]

5. [Maximum mark: 8]

Consider the curve C given by $y = x^3$.

The tangent at a point P on C meets the curve again at Q . The tangent at Q meets the curve again at R . Denote the x -coordinates of P , Q and R , by x_1, x_2 and x_3 respectively where $x_1 \neq 0$. Show that, x_1, x_2, x_3 form the first three elements of a divergent geometric sequence.

6. [Maximum mark: 8]

Consider the recurrence relation $H_{n+1} = 2H_n + 1$, $n \in \mathbb{Z}^+$ where $H_1 = 1$.

- (a) Find H_2, H_3 and H_4 . [2]
- (b) Conjecture a formula for H_n in terms of n , for $n \in \mathbb{Z}^+$. [1]
- (c) Prove by mathematical induction that your formula is valid for all $n \in \mathbb{Z}^+$. [5]

7. [Maximum mark: 9]

The function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f: x \rightarrow \begin{cases} -3x + 1 & \text{for } x < 0 \\ 1 & \text{for } x = 0. \\ 2x^2 - 3x + 1 & \text{for } x > 0 \end{cases}$

By considering limits prove that f is

- (a) continuous at $x = 0$; [4]
- (b) differentiable at $x = 0$. [5]

8. [Maximum mark: 14]

The points A, B have coordinates $(-3, 0), (5, 0)$ respectively. Consider the Apollonius circle C which is the locus of point P where

$$\frac{AP}{BP} = k \text{ for } k \neq 1.$$

Given that the centre of C has coordinates $(13, 0)$, find

- (a) (i) the value of k ;
(ii) the radius of C ;
(iii) the x -intercepts of C . [11]
- (b) Let M be any point on C and N be the x -intercept of C between A and B. Prove that $\widehat{AMN} = \widehat{NMB}$. [3]

9. [Maximum mark: 11]

- (a) Use the Euclidean algorithm to find $\gcd(162, 5982)$. [4]
- (b) The relation R is defined on \mathbb{Z}^+ by nRm if and only if $\gcd(n, m) = 2$.
- (i) By finding counterexamples show that R is neither reflexive nor transitive.
- (ii) Write down the set of solutions of $nR6$. [7]

10. [Maximum mark: 10]

- (a) Show that $2^n \equiv (-1)^n \pmod{3}$, where $n \in \mathbb{N}$. [3]
- (b) Hence show that an integer is divisible by 3 if and only if the difference between the sum of its binary (base 2) digits in even-numbered positions and the sum of its binary digits in odd-numbered positions is divisible by 3. [3]
- (c) Express the hexadecimal (base 16) number $ABBA_{16}$ in binary. [4]

11. [Maximum mark: 8]

The points P , Q and R , lie on the sides $[AB]$, $[AC]$ and $[BC]$, respectively, of the triangle ABC . The lines (AR) , (BQ) and (CP) are concurrent.

Use Ceva's theorem to prove that $[PQ]$ is parallel to $[BC]$ if and only if R is the midpoint of $[BC]$.

12. [Maximum mark: 14]

In this question, x , y and z denote the coordinates of a point in three-dimensional Euclidean space with respect to fixed rectangular axes with origin O . The vector space of position vectors relative to O is denoted by \mathbb{R}^3 .

(a) Explain why the set of position vectors of points whose coordinates satisfy $x - y - z = 1$ does not form a vector subspace of \mathbb{R}^3 . [1]

(b) (i) Show that the set of position vectors of points whose coordinates satisfy $x - y - z = 0$ forms a vector subspace, V , of \mathbb{R}^3 .

(ii) Determine an orthogonal basis for V of which one member is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(iii) Augment this basis with an orthogonal vector to form a basis for \mathbb{R}^3 .

(iv) Express the position vector of the point with coordinates $(4, 0, -2)$ as a linear combination of these basis vectors. [13]

13. [Maximum mark: 11]

The discrete random variables $X_n, n \in \mathbb{Z}^+$ have probability generating functions given by $G_n(t) = \frac{t}{n} \left(\frac{t^n - 1}{t - 1} \right)$.

(a) Use the formula for the sum of a finite geometric series to show that

$$P(X_n = k) = \begin{cases} \frac{1}{n} & \text{for } 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} . \quad [4]$$

(b) Find $E(X_n)$. [3]

Let X_{n-1} and X_{n+1} be independent.

(c) Find the set of values of n for which $E(X_{n-1} \times X_{n+1}) < 2n$. [4]

14. [Maximum mark: 16]

A matrix M is called idempotent if $M^2 = M$.

- (a) (i) Explain why M is a square matrix.
- (ii) Find the set of possible values of $\det(M)$. [4]

The idempotent matrix N has the form

$$N = \begin{pmatrix} a & -2a \\ a & -2a \end{pmatrix}$$

where $a \neq 0$.

- (b) (i) Find the value of a .
- (ii) Find the eigenvalues of N .
- (iii) Find corresponding eigenvectors. [12]
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